ON THE CANONICAL REAL STRUCTURE ON WONDERFUL VARIETIES

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Abstract. We study equivariant real structures on spherical varieties. We call such a structure canonical if it is equivariant with respect to the involution defining the split real form of the acting reductive group G. We prove the existence and uniqueness of a canonical structure for homogeneous spherical varieties G/H with H self-normalizing and for their wonderful embeddings. For a strict wonderful variety we give an estimate of the number of real form orbits on the set of real points.

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Introduction

A real structure on a complex manifold X is an anti-holomorphic involution $\mu : X \to X$. The set of fixed points X^μ of μ is called the real part of (X, μ). If it is clear from the context which μ is considered then the real part will be denoted by $\mathbb{R}X$. In our paper, we are interested in the algebraic case. This means that X is a complex algebraic variety, which we will assume non-singular though this is not needed for the

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definition of a real structure. Also, μ is algebraic in the sense that for any function f regular at $x \in X$ the function $\overline{f \circ \mu}$ is regular at $\mu(x)$.

It is not easy to classify all real structures on a given variety X. Much work is done for compact toric varieties, where one has the notion of a toric real structure. Namely, if X is a toric variety acted on by an algebraic torus T then a real structure $\mu : X \to X$ is said to be toric if μ normalizes the T-action. It is natural to classify toric real structures up to conjugation by toric automorphisms, i.e., by automorphisms of X normalizing the T-action. Again, such a classification is not easy. For toric surfaces and threefolds it was obtained by C. Delaunay; see [De].

In the toric case, there is a notion of a canonical real structure. This is a real structure which is usually defined as complex conjugation on the open T-orbit, but we prefer a slightly different and more general definition. Let $\sigma : T \to T$ be the involutive anti-holomorphic automorphism of the real Lie group T which coincides with inversion on the maximal compact torus $T_c \subset T$. If $T \simeq (\mathbb{C}^*)^n$ then the real form defined by σ is split, i.e., isomorphic to $(\mathbb{R}^*)^n$. A canonical real structure on a toric variety X is a real structure which satisfies

$$(*) \qquad \mu(a \cdot x) = \sigma(a) \cdot \mu(x)$$

for all $x \in X,\ a \in T$. Of course, a canonical real structure is uniquely defined by the image of one point in the open orbit and any two canonical real structures are related by $\mu'(x) = t \cdot \mu(x)$, where $t \in T$ and $\sigma(t) \cdot t = 1$.

Our goal is to generalize this notion to varieties acted on by reductive algebraic groups. Let G be a connected reductive algebraic group defined over $\mathbb{C}$. We recall that an algebraic involution θ of G is called a Weyl involution if $\theta(t) = t^{-1}$ for all t in some algebraic torus $T \subset G$. Such an involution is known to be unique up to conjugation by an inner automorphism. By Cartan Fixed Point Theorem, one can always find a maximal compact subgroup $K \subset G$, such that $\theta(K) = K$. Then the corresponding Cartan involution τ commutes with θ and the product $\sigma = \tau \circ \theta = \theta \circ \tau$ is an involutive anti-holomorphic automorphism of G defining the split real form. Assume now that G acts on an algebraic variety X. Then a real structure $\mu : X \to X$ is called canonical if μ satisfies the above condition $(*)$ for all $x \in X,\ a \in G$. We remark that it suffices to check $(*)$ only for $a \in K$, in which case one can replace σ by θ.

The most natural generalization of toric varieties to the case of reductive algebraic groups is the notion of spherical varieties, which we recall in Section 1.

Suppose X is affine and non-singular. For X spherical a canonical real structure $\mu : X \to X$ always exists ([A], Theorem 1.2). However, in the non-spherical case such a structure may not exist even if X is homogeneous ([AP], Proposition 6.3).

In this paper, we study the problem of existence of a canonical real structure for all homogeneous spherical varieties, affine or not affine. We also consider the similar question for some complete spherical varieties, namely the so-called wonderful varieties. The definition of wonderful and strict wonderful varieties is recalled in Section 1.

We start with a finiteness theorem for real structures on wonderful varieties (Theorem 2.4). Then we prove some topological properties of a canonical real structure on a wonderful variety provided such a structure exists (Theorem 3.10). After that we show that a canonical real structure exists and is uniquely defined for homogeneous spherical varieties G/H with H self-normalizing (Theorem 4.12) and for their wonderful completions (Theorem 4.13). As an application we show that for a spherical subgroup $H \subset G$, whose normalizer is self-normalizing, there is always an anti-holomorphic involution $\sigma : G \to G$, defining the split real form and such that $\sigma(H) = H$ (Theorem 4.14). Finally, we give an estimate of the total number of real form orbits in $\mathbb{R}X$ for the canonical real structure on a strict wonderful variety X (Theorem 5.19).

1. Wonderful varieties

Recall that G is a connected reductive algebraic group over $\mathbb{C}$. A normal algebraic G-variety X is called spherical if X contains an open orbit of a Borel subgroup $B \subset G$. We denote the open orbits of B and G on X by X_B° and X_G° respectively.

The following definition is due to D. Luna ([Lu1]). An algebraic G-variety X is called *wonderful* if

(i) X is complete and smooth;

(ii) X admits an open G-orbit whose complement consists of a finite union of smooth prime divisors $X_1 \dots, X_r$ with normal crossings;

(iii) the G-orbit closures of X are given by the partial intersections of the X_i's.

Remark that a wonderful variety X has a unique closed G-orbit. The latter is the full intersection of the boundary divisors X_i of X.

D. Luna proved that wonderful G-varieties are spherical. The connected center of G acts trivially on a wonderful variety, so if G acts effectively then G is semisimple. If a spherical homogeneous space G/H admits an equivariant wonderful embedding then such an embedding is unique up to a G-isomorphism; see [Lu1] and references therein.

By a theorem of F. Knop, a wondferful equivariant embedding of G/H always exists if the spherical subgroup H is self-normalizing in G; see [K].

Proposition 1.1. *Let X be a wonderful variety and let $X_G^\circ = G/H$. Then H has finite index in its normalizer.*

Proof. See Section 4.4 in [Br1]. □

A wonderful variety is called *strict* if each of its points has a self-normalizing stabilizer. The class of strict wonderful varieties includes flag varieties and De Concini-Procesi compactifications ([DP]). Strict wonderful varieties are classified in [BCF].

For any variety X, let $\mathrm{Aut}(X)$ denote the automorphism group of X. We will need the following proposition describing the identity component $\mathrm{Aut}_0(X)$ for a wonderful G-variety X.

Proposition 1.2 ([Br2], Theorem 2.4.2)**.** *If X is wonderful under G then $\mathrm{Aut}_0(X)$ is semisimple and X is wonderful under the action of $\mathrm{Aut}_0(X)$.*

In addition, we have the following proposition, for which we could not find a reference.

Proposition 1.3. *Let X be a wonderful variety. Then $\mathrm{Aut}_0(X)$ has finite index in $\mathrm{Aut}(X)$.*

Proof. Write $X_G^\circ = G/H$, where H is the stabilizer of a point $x_0 \in X_G^\circ$. Let N be the normalizer of H in G. By Proposition 1.1 the orbit $N \cdot x_0$ is finite. For any $\alpha \in \mathrm{Aut}(X)$ and $g \in \mathrm{Aut}_0(X)$ put

$$\iota_\alpha(g) = \alpha \cdot g \cdot \alpha^{-1}.$$

Let L denote the group of all automorphisms of the group $\mathrm{Aut}_0(X)$. Then we have the homomorphism

$$\varphi : \mathrm{Aut}(X) \to L, \quad \varphi(\alpha) = \iota_\alpha,$$

whose image contains the group of inner automorphisms of $\mathrm{Aut}_0(X)$. Since the latter group is semisimple by Proposition 1.2, $\mathrm{Im}(\varphi)$ has finitely many connected components. We now prove that $\mathrm{Ker}(\varphi)$ is finite. It then follows that $\mathrm{Aut}(X)$ has finitely many connected components.

If $\alpha \in \mathrm{Ker}(\varphi)$ then α commutes with all automorphisms from G. Thus $\alpha(gx_0) = g\alpha(x_0)$ for all $g \in G$. Since X has only one open G-orbit, we have $\alpha(X_G^\circ) = X_G^\circ$ and, in particular, $\alpha(x_0) = ax_0$ for some $a \in G$. Now take $g \in H$. Then

$$ax_0 = \alpha(x_0) = \alpha(gx_0) = g\alpha(x_0) = gax_0,$$

hence $aga^{-1} \in H$ and $a \in N$. Since the N-orbit of x_0 is finite, there are only finitely many possibilities for ax_0. But, for $\alpha(x_0)$ fixed, α is uniquely determined on the open G-orbit and thus everywhere on X. □

2. Finiteness theorem

The group $\mathrm{Aut}(X)$ acts on the set of real structures on X by

$$\mu \mapsto \alpha \cdot \mu \cdot \alpha^{-1}.$$

For X wonderful, we prove that this action has only finitely many orbits.

Theorem 2.4. *Let X be a wonderful variety. Then, up to an automorphism of X, there are only finitely many real structures on X.*

Proof. Assume that X has at least one real structure μ_0. Then $\mathrm{Aut}(X)$-orbits on the set of real structures on X are in one-to-one correspondence with the cohomology classes from $H^1(\mathbb{Z}_2, \mathrm{Aut}(X))$, where the generator $\gamma \in \mathbb{Z}_2$ acts on $\mathrm{Aut}(X)$ by sending α to $\mu_0\alpha\mu_0$. We now use the exact cohomology sequence, associated with the normal subgroup $\mathrm{Aut}_0(X) \triangleleft \mathrm{Aut}(X)$. From Corollary 3 in I.5.5 of [S] it follows that $H^1(\mathbb{Z}_2, \mathrm{Aut}(X))$ is finite if the following two conditions are fulfilled:

(1) $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is finite;

(2) for the $\mathbb{Z}_2$-action on $\mathrm{Aut}_0(X)$ obtained by twisting the given action by an arbitrary cocycle $a \in Z^1(\mathbb{Z}_2, \mathrm{Aut}(X))$ the corresponding cohomology set $H^1(\mathbb{Z}_2, {}_a\mathrm{Aut}_0(X))$ is finite.

We have just proved (1). Since $\mathrm{Aut}_0(X)$ is linear algebraic, (2) follows from Borel-Serre's Theorem (see [BS]). □

3. General properties of equivariant real structures

Let X be a non-singular complex algebraic variety with a real structure $\mu : X \to X$. Suppose G is a connected algebraic group acting on X and let $\sigma : G \to G$ be an involutive anti-holomorphic automorphism of G as a real algebraic group. Then the fixed point subgroup

$$G^\sigma = \{g \in G \mid \sigma(g) = g\}$$

is real algebraic and its identity component G_0^σ is a closed real Lie subgroup in G.

We call μ a σ-equivariant real structure if

$$\mu(g \cdot x) = \sigma(g) \cdot \mu(x) \quad \text{for all } g \in G,\ x \in X.$$

Later on, we will be interested in the case when G^σ is a split real form of a reductive group G; see Introduction. However, in the following elementary lemma G and σ are arbitrary.

Lemma 3.5. *Let $H \subset G$ be an algebraic subgroup and let $X = G/H$. Suppose $x_0 \in \mathbb{R}X$. Then the connected component of $\mathbb{R}X$ through x_0 coincides with $G_0^\sigma \cdot x_0$. The orbit $G_0^\sigma \cdot x_0$ is Zariski dense in X.*

Proof. Let n be the complex dimension of X. Then the real dimension of $\mathbb{R}X$ is also n. Since μ is σ-equivariant, we have $G^\sigma(x_0) \subset \mathbb{R}X$. Thus it suffices to show that $\dim G_0^\sigma(x_0) \geq n$. Let G_{x_0} be the stabilizer of x_0 in G. Then $G_0^\sigma \cap G_{x_0}$ is a totally real submanifold in G_{x_0}, hence

$$\dim_{\mathbb{R}} G_0^\sigma \cap G_{x_0} \leq \dim_{\mathbb{C}} G_{x_0},$$

and so we obtain

$$\dim G_0^\sigma \cdot x_0 = \dim G_0^\sigma - \dim G_0^\sigma \cap G_{x_0} \geq \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} G_{x_0} = n.$$

Finally, since $G_0^\sigma \cdot x_0 \subset X$ is a totally real submanifold of maximal possible dimension, $G_0^\sigma \cdot x_0$ is not contained in an algebraic subvariety of dimension smaller than n.

□

From now on G is reductive. We need some preparatory lemmas on the involution $\sigma : G \to G$ defining the split real form of G. We also fix some notation, which will be used all the time in the sequel. So let $T \subset G$ be a torus, on which σ acts as the involutive anti-holomorphic automorphism with fixed point subgroup being the non-compact real part of T. In coordinates, if $T \simeq (\mathbb{C}^*)^r$ then

$$\sigma(z_1, \ldots, z_r) = (\bar{z}_1, \ldots, \bar{z}_r), \quad z = (z_1, \ldots, z_r) \in (\mathbb{C}^*)^r.$$

Lemma 3.6. *Let χ be a character of T. Then $\overline{\chi \circ \sigma} = \chi$.*

Proof. Take t in the non-compact real part of T. Then $\sigma(t) = t$ and the value of χ is real. This shows that the weights $\overline{\chi \circ \sigma}$ and χ coincide on real points, hence also everywhere by analytic extension. □

Lemma 3.7. *Let $\mathfrak{g}$ be the Lie algebra of G. Denote the associated involution of $\mathfrak{g}$ again by σ. Then all root spaces in $\mathfrak{g}$ are σ-stable.*

Proof. Let $\alpha : T \to \mathbb{C}^*$ be a root, $\mathfrak{g}_\alpha$ the corresponding root space, $X_\alpha \in \mathfrak{g}_\alpha$, and $t \in T$. Then

$$\mathrm{Ad}(t) \cdot X_\alpha = \alpha(t) X_\alpha$$

implies

$$\mathrm{Ad}(\sigma(t)) \cdot \sigma(X_\alpha) = \overline{\alpha(t)}\, \sigma(X_\alpha)$$

or, equivalently,

$$\mathrm{Ad}(t)\cdot\sigma(X_\alpha) = \overline{\alpha\circ\sigma(t)}\,\sigma(X_\alpha) = \alpha(t)\,\sigma(X_\alpha),$$

where the last equality follows from Lemma 3.6. Therefore $\sigma(X_\alpha)\in\mathfrak{g}_\alpha$, showing that the root spaces are σ-stable. $\square$

Corollary 3.8. *With the above choice of T we have $\sigma(B) = B$ and $\sigma(P) = P$ for any Borel subgroup $B\subset G$ containing T and any parabolic subgroup $P\subset G$ containing B.*

Proof. The Lie algebras of $\mathfrak{p}$ and $\mathfrak{b}$ are spanned by root spaces and the Lie algebra $\mathfrak{t}$ of T, so Lemma 3.7 applies. $\square$

We will assume throughout the paper that T, B and P are chosen as in Corollary 3.8.

Proposition 3.9. *With the above choice of P, define a self-map of the flag variety $X = G/P$ by $\mu(g\cdot P) = \sigma(g)\cdot P$. Then μ is a σ-equivariant real structure on X. Such a structure is uniquely defined. The set $\mathbb{R}X$ is the unique closed G_0^σ-orbit on X. In particular, the (possibly disconnected) real form G^σ is transitive on $\mathbb{R}X$.*

Proof. Clearly, the map μ is correctly defined, anti-holomorphic, σ-equivariant, and involutive. If there is another σ-equivariant real structure μ' on X, then the product $\mu'\cdot\mu$ is an automorphism of X commuting with the G-action. Since P is self-normalizing, such an automorphism is the identity map, hence $\mu' = \mu$. By construction, the base point $e\cdot P$ is contained in $\mathbb{R}X$. According to Lemma 3.5, each connected component of $\mathbb{R}X$ is a closed G_0^σ-orbit. By [W] such an orbit is unique, so $\mathbb{R}X$ is connected and coincides with that orbit. The last assertion is now obvious. $\square$

For a wonderful variety X, the existence of a σ-equivariant real structure requires some work involving Luna-Vust invariants of spherical homogeneous spaces. We postpone this until the next section. Here, assuming that such a structure μ exists, we study geometric properties of $\mathbb{R}X$. The notation is as in Section 1. In particular,

$$Y = X_1\cap\ldots\cap X_r$$

is the unique closed G-orbit in X. Note that $\mu(Y) = Y$ and $\mathbb{R}X\cap Y$ is the unique closed G_0^σ-orbit in Y by Proposition 3.9.

Theorem 3.10. *Let X be any wonderful G-variety equipped with a σ-equivariant real structure μ. Then:*

(i) *G_0^σ has finitely many orbits on $\mathbb{R}X$;*

(ii) *$\mathbb{R}X\cap Gx\neq\emptyset$ for any $x\in X$;*

(iii) *there is exactly one closed G_0^σ-orbit in $\mathbb{R}X$; this orbit is contained in the closed G-orbit and is G^σ-homogeneous;*

(iv) $\mathbb{R}X$ *is connected.*

Proof. (i) $\mathbb{R}X$ is a non-empty real algebraic set. In particular, $\mathbb{R}X$ has finitely many connected components. By Lemma 3.5, each of them is one G_0^σ-orbit.

(ii) We choose B as in Corollary 3.8. We prove first that μ preserves G-orbits. This is clear for the open orbit, because its μ-image is also an open orbit which is unique. Since the orbit structure is well understood (see Section 1), it is enough to prove that $\mu(X_i) = X_i$, where X_i are the boundary divisors. Equivalently, it suffices to prove that the G-invariant valuation v_i centered on X_i is μ-invariant in the sense that

$$v_i(\overline{f \circ \mu}) = v_i(f)$$

for any $f \in \mathbb{C}(X) \setminus \{0\}$. It is enough to check this on B-eigenfunctions (see Appendix A.1), but then Lemma 3.6 yields the required equality.

Now, let $G \cdot x$ be any G-orbit on X and let $\mathrm{cl}(G \cdot x)$ be its Zariski closure within X. Since $G \cdot x$ is μ-stable, so is $\mathrm{cl}(G \cdot x)$. Note that $Y \subset \mathrm{cl}(G \cdot x)$, therefore $Z := \mathbb{R}X \cap \mathrm{cl}(G \cdot x) \neq \emptyset$. Furthermore, $\mathrm{cl}(G \cdot x)$ is a non-singular variety and $Z \subset \mathrm{cl}(G \cdot x)$ is a totally real submanifold of maximal possible dimension. Therefore Z is not contained in the boundary $\mathrm{cl}(G \cdot x) \setminus G \cdot x$.

(iii) Given a closed orbit $G_0^\sigma \cdot y \subset \mathbb{R}X$, consider the orbit $G^\sigma \cdot y$, which is also closed, and take a fixed point of the real form $B^\sigma \subset B$ thereon. The existence of such a point follows from Borel's theorem for connected split solvable groups. Assuming $B^\sigma \cdot y = y$, we also have $B \cdot y = y$. But then $G \cdot y$ is projective, i.e., $G \cdot y = Y$. Thus our statement is reduced to the case of flag varieties, and we can apply Proposition 3.9.

(iv) Assume $\mathbb{R}X$ is disconnected. Since $\mathbb{R}X \cap Y$ is connected, we can find a connected component W of $\mathbb{R}X$, such that $W \cap Y = \emptyset$. Then W is a closed G_0^σ-orbit and $G^\sigma \cdot W$ is a closed G^σ-orbit, which also has empty intersection with Y. On the other hand, by the above argument, B^σ has a fixed point on $G^\sigma \cdot W$. Since that fixed point is also fixed by B, it belongs to the closed G-orbit Y. We get a contradiction showing that $\mathbb{R}X$ is in fact connected.

□

4. The canonical real structure

Recall that T and B are chosen as in Corollary 3.8.

Proposition 4.11. *Any spherical subgroup $H \subset G$ is conjugate to $\sigma(H)$ by an inner automorphism of G.*

Proof. We note first that $\sigma(H)$ is a spherical subgroup of G. We shall prove that the Luna-Vust invariants attached to $X_1 = G/H$ and $X_2 = G/\sigma(H)$ are the same and then use the theorem in Appendix A.1, from where we also take the notations.

Consider the map $\mu : X_1 \to X_2$ defined by

$$X_1 \ni g \cdot H \overset{\mu}{\mapsto} \sigma(g) \cdot \sigma(H) \in X_2.$$

We show that μ defines a bijection between the sets of B-eigenfunctions on X_2 and X_1. Moreover, the associated map $\Lambda^+(X_2) \longrightarrow \Lambda^+(X_1)$ is the identity map. Namely, let f be a B-eigenfunction in $\mathbb{C}(X_2)$ and let λ be its B-weight. Then the complex conjugate of $f \circ \mu$ is a B-eigenfunction in $\mathbb{C}(X_1)$ with weight $\overline{\lambda \circ \sigma}$. The latter is equal to λ by Lemma 3.6. Since we can apply the same argument to the map μ^{-1}, it follows that the weight lattices of X_1 and X_2 coincide and μ induces the identity map on $\Lambda^+(X_2) = \Lambda^+(X_1)$.

Further, consider the map $\mathcal{V}(X_1) \to \mathcal{V}(X_2)$, defined by

$$v \mapsto (f \mapsto v(\overline{f \circ \mu})).$$

This map is obviously bijective. Namely, its inverse is defined analogously by means of the mapping $\mu^{-1} : X_2 \to X_1$.

Finally, there is a natural bijection $\iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$ sending D to $\pi_2[\sigma(\pi_1^{-1}(D))]$ where π_1 and π_2 are the projections from G to X_1 and X_2 respectively. For this mapping, $\varphi_{\iota(D)}$ evaluated on $\overline{\lambda \circ \sigma}$ gives the same result as φ_D evaluated on λ. By Lemma 3.6, $\overline{\lambda \circ \sigma}$ coincides with λ, and so we have $\varphi_{\iota(D)} = \varphi_D$. Similarly, $G_{\iota(D)} = \sigma(G_D) = G_D$ because G_D is a parabolic subgroup containing B. □

Theorem 4.12. *Let H be a spherical subgroup of G and $a \in G$ such that $\sigma(H) = aHa^{-1}$. The assignment*

$$\mu_0 : gH \mapsto \sigma(g)aH$$

defines an anti-holomorphic σ-equivariant diffeomorphism of G/H. If H is self-normalizing then this map is involutive, hence a σ-equivariant real structure on G/H. Furthermore, for H self-normalizing a σ-equivariant real structure on G/H is uniquely defined.

Proof. The first assertion follows from Proposition 4.11. Further, since σ is an involution of G, $\sigma(a)a$ belongs to the normalizer of H in G. The latter coincides with H. This proves the second assertion. The product of two σ-equivariant real structures on G/H is an automorphism of

G/H commuting with the G-action. For H self-normalizing in G such an automorphism is the identity map, and the last assertion follows. $\square$

Theorem 4.13. *Let H be a self-normalizing spherical subgroup of G and let X be the wonderful completion of G/H. Then there exists one and only one σ-equivariant real structure of X.*

Proof. Let $\iota : G/H \to X$ be the given wonderful completion and let $\bar{\iota} : G/H \to \bar{X}$ be the corresponding anti-holomorphic map with $\bar{X}$ being the complex conjugate of X. Recall that $\bar{X} = X$ as sets and that the sheafs of regular functions of $\bar{X}$ and X are complex conjugate.

We endow $\bar{X}$ with the G-action $(g, x) \mapsto \sigma(g) \cdot x$, where $(g, x) \mapsto g \cdot x$ is the given action of G on X. Note that this new action is regular on $\bar{X}$.

Consider the real structure μ_0 introduced in Theorem 4.12. Then $\bar{\iota} \circ \mu_0$ is again a wonderful completion of G/H. Since two wonderful completions of G/H are G-isomorphic, there exists a G-isomorphism $\mu : X \to \bar{X}$ such that $\mu \circ \iota = \bar{\iota} \circ \mu_0$. The map μ defines a σ-equivariant real structure on X.

Finally, a σ-equivariant real structure on X is defined by its restriction to the open G-orbit in X. The restriction is unique by Theorem 4.12. $\square$

In the remainder, the real structure defined in Theorem 4.13 is called *the canonical real structure of X*. We want to give here a group-theoretical application of Theorem 4.13.

Theorem 4.14. *If $H \subset G$ is a spherical subgroup with self-normalizing normalizer then there exists an anti-holomorphic involution $\sigma : G \to G$, defining the split real form and such that $\sigma(H) = H$. Moreover, one can find a Borel subgroup $B \subset G$, such that $B \cdot H$ is open in G and $\sigma(B) = B$.*

Proof. Let N be the normalizer of H in G. We start with some σ and take $a \in G$ as in Theorem 4.12, i.e., $\sigma(H) = aHa^{-1}$. Then, of course, $\sigma(N) = aNa^{-1}$. Let X be a wonderful equivariant completion of G/N and let μ be the canonical σ-equivariant real structure on X. By Theorem 3.10 we can find a μ-fixed point in the open orbit. Let $\mu(g_0 \cdot N) = g_0 \cdot N$. Replace σ by $\sigma_1 = i_{g_0}^{-1} \sigma i_{g_0}$, where i_{g_0} is the inner automorphism of G given by $x \mapsto g_0 x g_0^{-1}$. Also, replace μ by $\mu_1 = g_0^{-1} \mu g_0$. A straightforward calculation shows that $\mu_1(gx) = \sigma_1(g)\mu_1(x)$ for all $g \in G$, $x \in X$, i.e., μ_1 is a σ_1-equivariant real structure on X. Moreover, for the new pair (μ_1, σ_1) we have

$$\mu_1(e \cdot N) = (g_0^{-1} \mu g_0)(e \cdot N) = g_0^{-1} \mu(g_0 \cdot N) = e \cdot N.$$

Comparing the stabilizers at $e \cdot N$ and $\mu_1(e \cdot N)$, we get

$$\sigma_1(N) = N.$$

It follows that

$$N = \sigma_1(N) = i_{g_0}^{-1}\, \sigma\, i_{g_0}\,(N) = g_0^{-1}\sigma(g_0)\,\sigma(N)\,\sigma(g_0)^{-1}g_0.$$

As we have seen, $\sigma(N) = aNa^{-1}$. Substituting this in the previous equality, we get $g_0^{-1}\sigma(g_0)a \in N$, and it follows that $\sigma_1(H) = H$.

Now, assuming $\sigma(H) = H$ consider the subset $\Omega \subset G/B$ whose points correspond to the Borel subgroups $B_* \subset G$ with $B_* \cdot H$ open in G. Then Ω is Zariski open and σ-stable. The subset of σ-fixed Borel subgroups is a totally real submanifold in G/B, having maximal possible dimension. Thus its intersection with Ω is non-empty. □

Remark 4.15. The normalizer of a spherical subgroup is in general not self-normalizing, see Example 4 in [Av]. In Theorem 4.14, we do not know if the condition of N being self-normalizing is essential.

5. Real part: local structure and G_0^σ-orbits

Let X be a strict wonderful G-variety of rank r equipped with the canonical real structure μ. For a complex vector space V and an anti-linear map $\nu : V \to V$ we denote by the same letter ν the induced anti-holomorphic map of $\mathbb{P}(V)$.

Proposition 5.16. *There exist a simple G-module V with the associated representation $\rho : G \to \mathrm{GL}(V)$, an anti-linear involutive map $\nu : V \to V$, and an embedding $\varphi : X \to \mathbb{P}(V)$, such that*

(i) $\nu(\rho(g) \cdot v) = \rho(\sigma(g)) \cdot \nu(v) \quad (v \in V)$,

(ii) $\varphi(gx) = \rho(g) \cdot \varphi(x) \quad (x \in X)$ *and*

(iii) $\varphi(\mu x) = \nu\varphi(x) \quad (x \in X)$.

In particular, $\mathbb{R}X$ is G^σ-equivariantly embedded into the real projective space $\mathbb{RP}(V) \subset \mathbb{P}(V)$, defined by ν.

Proof. Since X is a non-singular projective G-variety, X can be G-equivariantly embedded into the projectivization of a G-module. Let $\varphi : X \to \mathbb{P}(V)$ be such an embedding and let $\rho : G \to \mathrm{GL}(V)$ denote the representation associated to V. Since X is a strict wonderful variety, we may choose V to be simple; see [P].

Now, equip the complex conjugate vector space $\bar{V}$ with the G-module structure given by $g \mapsto \overline{\rho(\sigma(g))}$. By Lemma 3.6, it follows that the G-modules V and $\bar{V}$ are isomorphic. In other words, we have an anti-linear map $\nu : V \to V$ satisfying (i). Though ν is not necessarily involutive, we can modify ν to get this property. As in Appendix A.2,

let v^- be a lowest weight vector of V. Then $\nu(v^-)$ is also a lowest weight vector, hence $\nu(v^-) = cv^-$ for some $c \in \mathbb{C}^*$. This implies $\nu^2(v^-) = \nu(cv^-) = \bar{c} \cdot \nu(v^-) = |c|^2 v^-$. Replacing ν by $\nu/|c|$, we get an involutive anti-linear map satisfying (i).

Since (ii) is clear from the construction, it remains to show (iii). Note that $\nu \circ \varphi \circ \mu$ is another G-equivariant embedding of X into $\mathbb{P}(V)$. Thus (iii) follows from the uniqueness of such an embedding; see [P]. $\square$

Let Z be the slice defined in Appendix A.2. We show that Z can be chosen to be μ-stable. As we have seen in Proposition 5.16, the line $\mathbb{C} \cdot v^-$ is ν-stable. So we may assume that $\nu(v^-) = v^-$. Then the tangent space $W := T_{v^-} G \cdot v^-$ is also ν-stable. Consider the real vector space $\mathbb{R}V = \{v \in V \mid \nu(v) = v\}$ and let $\mathbb{R}W = W \cap \mathbb{R}V$. Then $\mathbb{R}W$ is stable under L^σ and, also, under the Lie algebra $\mathfrak{l}^\sigma$ of L^σ. Now, the center of $\mathfrak{l}^\sigma$ is contained in the center of the complexified algebra $\mathfrak{l} = \mathfrak{l}^\sigma \otimes \mathbb{C}$ and is therefore represented by semisimple endomorphisms of $\mathbb{R}V$. The complete reducibility theorem for reductive Lie algebras over $\mathbb{R}$ implies that $\mathbb{R}W$ has a $\mathfrak{l}^\sigma$-stable complement in $\mathbb{R}V$; see [C], Ch.IV, § 4. Call this complement $E_\mathbb{R}$. The complexification $E_\mathbb{R} \otimes \mathbb{C} \subset V$ is $\mathfrak{l}$-stable and therefore L-stable. So we can take $E = E_\mathbb{R} \otimes \mathbb{C}$. Note that $E_\mathbb{R} = E \cap \mathbb{R}V$ is not just $\mathfrak{l}^\sigma$-stable, but also L^σ-stable even if L^σ is disconnected.

Obviously, $\nu(E) = E$. Furthermore, the linear form η in Appendix A.2 can be chosen real. Therefore, using (iii) of Proposition 5.16, we see that $\mu(Z) = Z$. Note that $P^u \cdot Z$ is μ-stable and $\mathbb{R}(P^u \cdot Z) = (P^u)^\sigma \cdot \mathbb{R}Z$.

The first assertion of the following proposition is a real analogue of Local Structure Theorem in [BLV]; see also Appendix A.2.

Proposition 5.17. (i) *The natural mapping*

$$(P^u)^\sigma \times (\mathbb{R}Z) \to (P^u)^\sigma \cdot \mathbb{R}Z = \mathbb{R}(P^u \cdot Z)$$

is a $(P^u)^\sigma$-equivariant isomorphism.

(ii) *Each G_0^σ-orbit in $\mathbb{R}X$ contains points of the slice Z.*

Proof. The first assertion follows readily from Local Structure Theorem and the above construction of Z.

To prove (ii), take a point $x \in \mathbb{R}X$. Since X is wonderful, the orbit $G \cdot x$ is not contained in a prime divisor $D \in \mathcal{D}(X)$. The intersection $G \cdot x \cap \left(\cup_{\mathcal{D}(X)} D\right)$ is a proper Zariski closed subset in $G \cdot x$. By the last assertion of Lemma 3.5, this subset does not contain $G_0^\sigma \cdot x$. Thus $G_0^\sigma \cdot x \cap X \setminus \cup_{\mathcal{D}(X)} D \neq \emptyset$, and (ii) follows from (i). $\square$

In the remainder, x denotes a real point in $X_G^\circ \cap Z$ and $H \subset G$ is the stabilizer of x. We assume that $\sigma(B) = B$ and $\mu(Z) = Z$. It follows that $\sigma(H) = H$. Note also that $B \cdot H$ is open in G because the orbit $B \cdot x$ is open in X.

As we recall in Appendix A.2, T acts linearly on Z and the corresponding characters, say $\gamma_1, \ldots, \gamma_r$, are linearly independent. These characters are usually called *spherical roots of* X. Further, we have

$$T \cap H = \bigcap_i \ker \gamma_i.$$

Set

$$A = T/T \cap H$$

and let ${}_2A \subset A$ be the subgroup of elements of order at most 2. Note that any element $t \in T$ can be uniquely written as

$$t = t_0 t_1, \quad \text{where } t_0 \in T_0^\sigma \text{ and } \sigma(t_1) = t_1^{-1}.$$

Such a decomposition of t will be referred to as the decomposition of t with respect to σ.

Proposition 5.18. *The T_0^σ-orbits of $\mathbb{R}Z \cap X_G^\circ$ are in one-to-one correspondence with the elements of ${}_2A$. In particular, the number of such orbits does not exceed 2^r.*

Proof. Let $t \in T$ and let $y = t \cdot x$ be a real point. Then $(\gamma_i \circ \sigma)(t) = \gamma_i(t)$ for every spherical root γ_i of X. By Lemma 3.6, it follows that $\gamma_i(t)$ is real-valued. If $t = t_0 t_1$ is the decomposition of t with respect to σ, then we have $\gamma_i(t_1) = \pm 1$. Therefore $t_1^2 \in H$. Assigning to $y \in \mathbb{R}Z \cap X_G^\circ$ the image of t_1 in A, we get a correctly defined map from the set of T_0^σ-orbits on $\mathbb{R}Z \cap X_G^\circ$ to ${}_2A$:

$$\alpha : T_0^\sigma \setminus (\mathbb{R}Z \cap X_G^\circ) \ \to \ {}_2A.$$

The injectivity of α is obvious. To prove the surjectivity, take any $t \in T$, such that $t^2 \in H$. Then $\gamma_i(t^2) = 1$, hence $\gamma_i(t) = \pm 1$. It follows that $t \cdot x$ is a real point. Furthermore, $\gamma_i(t_0) = 1$ and $\gamma_i(t_1) = \gamma_i(t)$. Hence $t \cdot H = t_1 \cdot H$ and $\alpha(T_0^\sigma \cdot x) = t \bmod T \cap H$. $\square$

Let I denote a subset of $\{1, \ldots, r\}$ and let $O_I \subset X$ be the corresponding G-orbit. Recall that O_I is μ-stable.

Theorem 5.19. (i) *Each G_0^σ-orbit in $\mathbb{R}O_I$ intersects the slice Z in a finite number of T_0^σ-orbits. The number of T_0^σ-orbits in $\mathbb{R}Z \cap O_I$ does not exceed $2^{r-|I|}$.*

(ii) $\mathbb{R}O_I$ *contains at most $2^{r-|I|}$ G^σ-orbits.*

(iii) *The total number of G_0^σ-orbits in $\mathbb{R}X$ is smaller than or equal to*
$$\sum_{k=0}^{r} 2^k \binom{r}{k}.$$

Proof. Recall that the G-orbit closures in X are also strict wonderful G-varieties. Furthermore, the rank of the orbit closure $\mathrm{cl}(O_I)$ equals $r - |I|$; see Sect. 3.2 in [Lu2]. Thus (i) follows from (ii) of Proposition 5.17, along with the estimate in Proposition 5.18. From (i) we get (ii), and (iii) is obtained by summing up over all G-orbits. $\square$

Example 1. Let $(\mathbb{P}^n)^*$ denote the variety of hyperplanes of $\mathbb{C}^{n+1}$ and let $X = \mathbb{P}^n \times (\mathbb{P}^n)^*$ be acted on diagonally by $G = PGL_{n+1}(\mathbb{C})$. Suppose $n > 1$. Then X is a strict wonderful variety of rank 1. The canonical real structure μ is defined by the complex conjugation on each factor of X. Moreover, $G_0^\sigma = G^\sigma = PGL_{n+1}(\mathbb{R})$ acts on $\mathbb{R}X$ with two orbits.

Example 2. Consider the quadratic form
$$F(z) = z_1^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_{p+q}^2, \quad q \geq p > 0, \quad p + q > 2.$$
The corresponding orthogonal group $G = SO_F$ acts on $X = \mathbb{P}^m$ as a subgroup of $\mathrm{SL}_{m+1}(\mathbb{C})$, where $m = p + q - 1$. Under this action, X is a two-orbit G-variety. The closed G-orbit is given by the equation $F = 0$. Again, X is a strict wonderful variety of rank 1. Let $\mu : X \to X$ and $\sigma : G \to G$ be the involutive mappings defined by complex conjugation. Then μ is a σ-equivariant real structure on X. Note that σ defines a split real form of G only for $q = p$ or $q = p + 1$. The real part $\mathbb{R}X$ is the real projective space $\mathbb{RP}^m$, on which G_0^σ acts with three orbits: $F > 0$, $F < 0$ and $F = 0$.

Remark 5.20. Starting with a real semisimple symmetric space, A. Borel and L. Ji considered the wonderful completion of the complexified homogeneous space. In this special setting, the completion is defined over $\mathbb{R}$ in a natural way. For the description of real group orbits on the set of its real points see [BJ], chapters 5 - 7.

Appendix A. Spherical varieties: invariants and local structure

A.1. **Luna-Vust invariants of spherical homogeneous spaces.** We recollect the definition of the combinatorial invariants attached to a given spherical G-variety X; see [LV].

Let $\mathbb{C}(X)$ denote the function field of X. Then the natural left action of G on X yields a G-module structure on $\mathbb{C}(X)$. *The weight lattice*

$\Lambda^+(X)$ is the set of B-weights of the B-eigenfunctions of $\mathbb{C}(X)$. Since X is spherical, the χ-weight space of $\mathbb{C}(X)$ is of dimension 1 for every $\chi \in \Lambda^+(X)$.

Let $\mathcal{V}(X)$ be the set of G-invariant discrete $\mathbb{Q}$-valued valuations of $\mathbb{C}(X)$. Consider the mapping

$$\rho : \mathcal{V}(X) \to \mathrm{Hom}(\Lambda^+(X), \mathbb{Q}), \quad v \mapsto (\chi \mapsto v(f_\chi)).$$

where f_χ is a B-eigenfunction of $\mathbb{C}(X)$ of weight χ. The map ρ is injective, hence one may regard $\mathcal{V}(X)$ in $\mathrm{Hom}(\Lambda^+(X), \mathbb{Q})$. Further, this cone is convex and simplicial. The cone $\mathcal{V}(X)$ is called *the valuation cone of* X; see for instance [Br1].

Define *the set of colors* $\mathcal{D}(X)$ of X as the set of B-stable, but not G-stable prime divisors of X. This is a finite set equipped with two maps, namely, $D \mapsto \rho(v_D)$ and $D \mapsto G_D$ with v_D (resp. G_D) being the valuation defined by (resp. the stabilizer in G of) the color D.

The Luna-Vust invariants of X are given by the triple $\Lambda^+(X)$, $\mathcal{V}(X)$, $\mathcal{D}(X)$. For two spherical G-varieties X and X', the equality $\mathcal{D}(X) = \mathcal{D}(X')$ means that there exists a bijection $\iota : \mathcal{D}(X) \to \mathcal{D}(X')$, such that $G_D = G_{\iota(D)}$ and $\rho(v_D) = \rho(v_{\iota(D)})$.

Theorem 1.21 ([Lo])**.** *Let H and H' be spherical subgroups of G. If H and H' have the same Luna-Vust invariants then they are G-conjugate.*

A.2. **Local structure.** We recall the so-called Local Structure Theorem with special emphasis on the case of wonderful varieties.

Let G denote a connected reductive algebraic group. Fix a Borel subgroup B of G and a maximal torus $T \subset B$ of G.

First, consider any normal and irreducible G-variety X and let Y be a complete G-orbit of X. Let $y \in X$ be fixed by the Borel subgroup B^- of G opposite to B and containing T. Let P denote the parabolic subgroup of G opposite to the stabilizer G_y and containing T. Then $L = P \cap G_y$ is a Levi subgroup of P, so that $P = L \cdot P^u$, where P^u is the unipotent radical of P. Theorem 1.4 in [BLV] asserts that there exists an affine L-variety Z, such that $y \in Z$ and the natural map $P^u \times Z \to P^u \cdot Z$ is an isomorphism.

Suppose now that X is spherical and denote the set of colors of X by $\mathcal{D}(X)$; see Appendix A.1. Further, if Y is the unique closed G-orbit in X then $P^u Z$ is the affine set $X \setminus \cup_{\mathcal{D}(X)} D$ and Z is a spherical L-variety. In the case of wonderful varieties Theorem 1.4 in [BLV] can be formulated as follows; see [Lu1], Sect. 1.1, 1.2, and [Br1], Sect. 2.2 - 2.4.

Theorem 1.22. *Assume X is a wonderful G-variety. There exists an affine L-subvariety Z of X containing y such that*

(i) *$P^u \times Z \to X \setminus \cup_{\mathcal{D}(X)} D : (p, z) \mapsto p.z$ is an isomorphism.*

(ii) *The derived group of L acts trivially on Z and Z intersects each G-orbit of X in one single T-orbit.*

(iii) *The variety Z is the affine space of dimension equal to the rank r of X Moreover, Z is acted on linearly by T and the corresponding r characters of T are linearly independent.*

Note that (ii) is a consequence of the configuration of the G-orbit closures in a wonderful variety. To obtain (iii), remark that Z is smooth since so is X and thereafter apply (ii) together with Luna Slice Theorem.

Let us now recollect how the slice Z is constructed in the case of strict wonderful varieties. One may consult [BLV] for a general treatment. Let

$$\varphi : X \to \mathbb{P}(V)$$

be an embedding of X within the projectivization of a finite dimensional G-module V. Thanks to [P], we can take V to be simple. Then y regarded in $\mathbb{P}(V)$ can be written as $[v^-]$ with v^- being a B^--eigenvector of V (unique up to a scalar). Since L is reductive, there exists an L-module submodule $E \subset V$ such that

$$V = T_{v^-} G \cdot v^- \oplus E,$$

where $T_{v^-} G \cdot v^-$ stands for the tangent space of the orbit $G \cdot v^-$ at the point v^-. Let η be the linear form on V such that $\eta(v^-) = 1$ and η is a B-eigenvector. Let $\mathbb{P}(\mathbb{C}v^- \oplus E)_\eta$ be the open set of $\mathbb{P}(V)$ on which η does not vanish. Then

$$Z = \varphi^{-1}\left(\mathbb{P}(\mathbb{C}v^- \oplus E)_\eta\right).$$

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